

II

THE PRINCIPLE OF SUFFICIENT REASON

AN outstanding characteristic of present-day civilization is the extraordinary rapidity of scientific advance, accompanied by a veritable Babel of changing scientific theories. The ceaseless flux has produced confusion in the minds of men. Even philosophy is affected by the prevailing uncertainties, and many assert that its speculations are meaningless unless narrowly restrained to the mathematical and logical fields.

But philosophy has an answer ready for these detractors. For it has been Plato, Aristotle, Leibniz, and other philosophers who first emphasized the basic significance of mathematics and logic; and it may be reasonably conjectured that in the future, seers endowed with the requisite philosophic insight will again open up large vistas which the specialized scientist, mathematician, and logician are likely to miss. Only those are likely to contribute in this way who, like Plato, Aristotle, and Leibniz, have mastered the essence of contemporary scientific knowledge. Unfortunately the task of achieving this necessary synthesis is becoming a more and more difficult one.

It will be recalled that Plato had inscribed at the portals of his famous Academy, "Let no one ignorant of Geometry enter here." His whole philosophic system affirmed the supreme importance of mathematical thought and of abstract Ideas generally, of which the concrete instances to be found in the actual world were held to be merely inferior copies.

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His contemporary, Aristotle, was the creator of formal logic.

The German mathematician and philosopher Leibniz has been regarded by some as Plato's true successor in modern times. He too, emphasized the fundamental rôle of abstract thought for philosophy, and foresaw the possibility of a logical calculus such as is realized in modern symbolic logic. He said, "My metaphysics is all mathematics" and even went so far as to declare that "The mathematicians have as much need of being philosophers as the philosophers of being mathematicians."

At the foundation of Leibniz's philosophy were two logical principles to which he attached the greatest importance: the Principle of Contradiction and the Principle of Sufficient Reason. Both seem to have been explicitly adopted by him as early as the year 1686. The first principle merely asserts that every proposition is either true or false—*tertium non datur*. Nearly everyone would admit without hesitation that this is a valid principle if not indeed a truism; and yet, strangely enough, its validity has been denied in the intuitionist logic of the Dutch mathematician Brouwer, who allows, as a kind of intermediate category, the proposition which admits neither of proof nor disproof. It is the second principle, in modified form, to which I wish to direct attention.

It is desirable to recall in the first place Leibniz's own formulation of the Principle of Sufficient Reason,¹ "that every true proposition, which is not known *per se*, has an *a priori* proof, or, that a reason can be given for every truth, or, as is commonly said, that nothing happens without a reason." Leibniz adds that "Arithmetic and Geometry do not need this Principle, but Physics and Mechanics do, and

¹The translations here used are those of Bertrand Russell, "A Critical Exposition of the Philosophy of Leibniz," (Cambridge, England, 1900).

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Archimedes employed it.” Furthermore he used the same principle for metaphysical purposes as the following remarkable conclusion, obviously based on this Principle, shows: “When two things which cannot both be together, are equally good; and neither in themselves nor by their combination with other things, has the one any advantage over the other: God will produce neither of them.”

My primary purpose will be to show how a properly formulated Principle of Sufficient Reason plays a fundamental rôle in scientific thought and, furthermore, is to be regarded as of the greatest suggestiveness from the philosophic point of view.²

In the preceding lecture I pointed out that three branches of philosophy, namely Logic, Aesthetics, and Ethics, fall more and more under the sway of mathematical methods. Today I would make a similar claim that the other great branch of philosophy, Metaphysics, in so far as it possesses a substantial core, is likely to undergo a similar fate. My basis for this claim will be that metaphysical reasoning always relies on the Principle of Sufficient Reason, and that the true meaning of this Principle is to be found in the “Theory of Ambiguity” and in the associated mathematical “Theory of Groups.”

If I were a Leibnizian mystic, believing in his “pre-established harmony,” and the “best possible world” so satirized by Voltaire in “Candide,” I would say that the metaphysical importance of the Principle of Sufficient Reason and the cognate Theory of Groups arises from the fact that God thinks multi-dimensionally³ whereas men can only think in linear syllogistic series, and the Theory of Groups is

²As far as I am aware, only Scholastic Philosophy has fully recognized and exploited this principle as one of basic importance for philosophic thought.

³That is, uses multi-dimensional symbols beyond our grasp.

the appropriate instrument of thought to remedy our deficiency in this respect.

The founder of the Theory of Groups was the mathematician Evariste Galois. At the end of a long letter written in 1832 on the eve of a fatal duel, to his friend Auguste Chevalier, the youthful Galois said in summarizing his mathematical work,⁴ "You know, my dear Auguste, that these subjects are not the only ones which I have explored. My chief meditations for a considerable time have been directed towards the application to transcendental Analysis of the theory of ambiguity. . . . But I have not the time, and my ideas are not yet well developed in this field, which is immense." This passage shows how in Galois's mind the Theory of Groups and the Theory of Ambiguity were interrelated.⁵

Unfortunately later students of the Theory of Groups have all too frequently forgotten that, philosophically speaking, the subject remains neither more nor less than the Theory of Ambiguity. In the limits of this lecture it is only possible to elucidate by an elementary example the idea of a group and of the associated ambiguity.

Consider a uniform square tile which is placed over a marked equal square on a table. Evidently it is then impossible to determine without further inspection which one of four positions the tile occupies. In fact, if we designate its vertices in order by A, B, C, D , and mark the corresponding positions on the table, the four possibilities are for the corners A, B, C, D of the tile to appear respectively in the positions A, B, C, D ; B, C, D, A ; C, D, A, B ; and D, A, B, C . These are obtained respectively from the first position by a

⁴My translation.

⁵It is of interest to recall that Leibniz was interested in ambiguity to the extent of using a special notation \vee (Latin, *vel*) for "or." Thus the ambiguously defined roots 1, 5 of $x^2 - 6x + 5 = 0$ would be written $x = 1 \vee 5$ by him.

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null rotation (I), by a rotation through 90° (R), by a rotation through 180° (S), and by a rotation through 270° (T). Furthermore the combination of any two of these rotations in succession gives another such rotation. Thus a rotation R through 90° followed by a rotation S through 180° is equivalent to a single rotation T through 270° , i.e., $RS = T$. Consequently, the "group" of four operations I , R , S , T has the "multiplication table" shown here:

	I	R	S	T
I	I	R	S	T
R	R	S	T	I
S	S	T	I	R
T	T	I	R	S

This table fully characterizes the group, and shows the exact nature of the underlying ambiguity of position.

More generally, any collection of operations such that the resultant of any two performed in succession is one of them, while there is always some operation which undoes what any operation does, forms a "group."

In the discussion of our main topic we turn first to some illustrations of the Principle of Sufficient Reason in the mathematical and physical domains. In the biological and psychological domains there appears to be as yet no scope for the Principle, due perhaps to the fact that our knowledge is not yet deep enough in these more complicated fields; however, it will be seen that in a certain metaphysical sense, the Principle is already of speculative importance for biological thought. In the social domain it will be possible to

point out how the Principle plays a decisive rôle in certain special problems; but here our knowledge is still less. Finally, I shall attempt a general quasi-mathematical formulation of the Principle itself, and venture a heuristic conjecture as to its ultimate rôle in scientific and philosophic thought.

Let us begin with some simple remarks concerning the nature of the ordinary complex number-system which, of course, is the most important technical weapon of mathematics. In the first place, we may note that there is a kind of similarity between the processes of addition and of multiplication, which is not so complete as to give rise to essential ambiguity, but yet is extensive enough to permit of the invention of logarithms replacing multiplication by addition. Secondly, there is a thorough-going similarity between the relations of "greater than" ($>$) and "less than" ($<$), holding between pairs of numbers. But, more significantly still, the number-system is finally regarded as completed by the inclusion of the "imaginary unit," $i = \sqrt{-1}$; and this unit is essentially ambiguous in that one might equally well write $i = -\sqrt{-1}$. In either case we have the basic equation $i^2 = -1$. Here the underlying group has only two operations, A , B , respectively taking i into itself and into its negative, with the multiplication table:

	A	B
A	A	B
B	B	A

This ambiguity, inherent in our number system, affords a very simple and mathematically unexceptionable application of the Principle of Sufficient Reason in the following way.

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Suppose that there be given an algebraic equation with real coefficients (as for instance $x^2 - 2x + 2 = 0$), with one imaginary root, $a + bi$ ($1 + i$ for the quadratic equation just mentioned). Then *without further argument*, on the basis of our Principle alone, it follows that $a - bi$ ($1 - i$ in the special case) is also a root of the equation. For why should $a + bi$ be a root any more than $a - bi$? The usual proof that $a - bi$ is also a root, involves the property of a complex number $a + bi$, that, if it vanishes, both the real coefficient a and the imaginary coefficient b vanish separately.

This example is so illuminating that it is worth while to state the underlying argument based on the Principle a little more fully:

(1) The ordinary complex number system is regarded as given, and contains the imaginary unit i ; and it is observed that if i be replaced by $-i$ the set of underlying postulates still holds, with the modified imaginary unit $-i$;

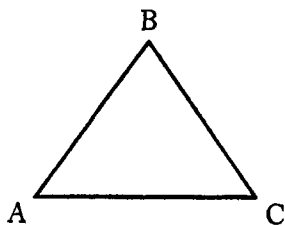
(2) It is given that $x = a + bi$ is a root of an algebraic equation with real coefficients $f(x) = 0$, i.e., $f(a + bi) = 0$;

(3) Changing i to $-i$ throughout, we infer without further argument that $f(a - bi) = 0$, in other words that $a - bi$ is also a root. It is to be noted that, since the coefficients of the equation are *real*, the equation itself is not altered when i is replaced by $-i$.

There is also a kindred suggestive general algebraic principle, that when numbers are algebraically defined in an unambiguous manner, they are *rationally* known, i.e., are given by ordinary fractions. It was by means of the extension of this last principle to ambiguously-defined algebraic quantities (e.g., as $\sqrt{2}$, ambiguously defined by $x^2 = 2$), and the related theory of Permutation Groups that Galois was able to formulate a definitive theory of the solution of algebraic equations by radicals.

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A more important use of the Principle of Sufficient Reason is afforded by elementary Euclidean geometry. Here the "group of motions" is the one which is important. The operations of this group move geometrical figures in space without changing their dimensions, and thus allow us to determine the congruence of figures by superposition. As a very simple illustration, consider an isosceles triangle ABC with given $\angle ABC$, and given equal sides $BA = BC$. Clearly the Principle of Sufficient Reason indicates that



the base angle, $\angle BAC$, is equal to the other base angle, $\angle BCA$. For why should it be greater rather than less? In fact, if we turn the triangle over so that BC takes the position BA , then BA must fall along the line BC , while the point A must coincide with C . But, philosophically speaking, the gist of the argument depends on the Principle of Sufficient Reason rather than upon a use of superposition.

Now there is inherent in geometry a more extensive ambiguity than that of *congruent* figures, namely that of *similar* figures. According to Couturat⁶ the corresponding larger group of similitudes is really the one appropriate to an approach to geometry from the standpoint of intuition (Sufficient Reason).

In Leibniz's own attempt to construct a "geometric calculus" of position, he makes various uses of the group of similitudes, but finally abandons it in favor of the Euclidean

⁶L. Couturat, *La Logique de Leibniz* (Paris, 1901), pp. 410-416.

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group of motions. Couturat believes that this was the basic reason for Leibniz's lack of success in his geometric calculus. Nevertheless Leibniz's attempt was of the highest interest: for Boole in his *Algebra of Logic* and Grassmann in his *Calculus of Extension* "later justified the most daring conceptions of Leibniz, in showing that these were not dreams, but prophetic intuitions which anticipated by nearly two centuries the progress of science and the human spirit."⁷

Possibly a simple system of postulates of my own for geometry⁸ is nearest in spirit to the suggestion of Couturat's, for my approach is based on the group of translations (scale), the group of rotations (protractor), and the group of expansions of figures about a fixed point (pantograph), and these are the three principal "subgroups" making up the group of similitudes in the plane.

Indeed, in any mathematical approach to geometry, it will be found that the choice of postulates is largely determined by considerations based on an underlying group, whether that of motions as in Euclid or of similitudes as suggested by Couturat, or of some other group like the so-called projective group.

It may be remarked in passing that for the primitive mind unacquainted with formal geometrical truth, the form of the circle, as embodied in the full moon, for instance, must have symbolized somehow the essential characteristics of the group of rotations about the center of the circle and of the associated ambiguity of all directions through the center.

It is interesting that Leibniz himself had the erroneous belief that, with a more thoroughgoing use of his Principles of Contradiction and of Sufficient Reason, it would be possible to prove the familiar axioms of Euclid. Thus he says: "Far

⁷My translation of Couturat.

⁸*Annals of Mathematics*, vol. 33 (1932), pp. 329-345.

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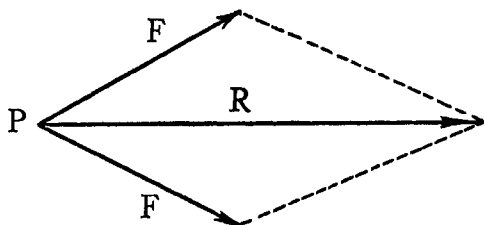
from approving the acceptance of doubtful principles, I would have people seek even the demonstration of the axioms of Euclid. . . . And when I am asked the means of knowing and examining innate principles I reply . . . that . . . we must try to reduce them to first principles, i.e., to axioms which are identical or immediate by means of definitions which are nothing but a distinct exposition of ideas.”

Another very important mathematical subject in which the Principle of Sufficient Reason enters is that of the Theory of Probability. If a coin is tossed, it is regarded as equally probable that the coin fall heads or tails in the successive throws. For how can the irrelevant marking on the two sides of the coin make either side more likely than the other? It is clear that the weightiest practical decisions must generally be guided by similar estimates of probability, whether these be in the successful conduct of the affairs of a nation, a business house, or an individual.

It is, however, in physics that one finds still more remarkable instances of the application of the Principle of Sufficient Reason. Only a few of these can be sketched here, of course.

We will begin with the case of two “equal forces” F acting upon a point P .⁹ If we admit that these are equivalent to a single resultant force R , then it is obvious from our Principle that the resultant must lie in the plane of the two forces. For why should it lie upon one side of this plane rather than upon the other? Here the tacit assumption is that the associated group is once more the group of motions characteristic of ordinary geometry. It is clear, for the same reason, that this resultant must fall along the internal bisector of the lines of the two forces (see the figure).

⁹A similar application is to a balance with equal weights in the two pans of the balance.



Furthermore, by a more elaborate but valid use of the Principle of Sufficient Reason, we may prove that in general the resultant of any two forces *must* be that given by the familiar Parallelogram Law¹⁰. This is the type of “proof” used for instance by the French astronomer and mathematician Laplace in his *Mécanique Céleste*.

Much of classical mechanics may be approached in a similar spirit. For example, the attracting gravitational force acting between two point masses must evidently lie along the line joining the two masses, by the same Principle; and the forces of action and reaction must be equal—for why should one force be greater than the other?—and there can be no absolute unit of length—for why is one unit to be preferred rather than another? Thus it follows from the Principle that the attractive force varies as some power of the distance.

If now we further impose the (philosophically reasonable) requirement that the mutual interaction is small at large distances apart, we find that the force must vary as some negative power of the distance. Thus we are led practically to the inverse first or second powers by use of the Principle of Sufficient Reason alone; and the planetary laws of Kepler indicate at once that the inverse second power of Newton has to be chosen. Thus from the standpoint of the Principle of Sufficient Reason, based on the ambiguity of the group of

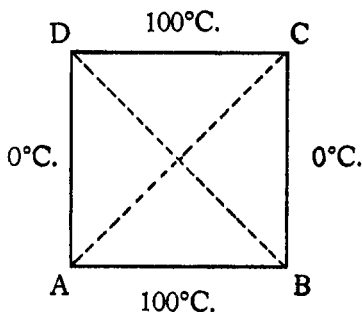
¹⁰See the Note appended to this lecture.

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motions and similitudes, the gravitational law of Newton appears as *almost* inevitable.

It is worthy of remark in this connection that Leibniz, searching for the invariable “substances” underlying the changing appearance of mechanical systems such as the solar system, was led to formulate the laws of the conservation of momentum, and of energy or *vis viva*, so important for modern physics.

Another instructive illustration of the power of the Principle of Sufficient Reason in certain physical questions is furnished by the following special problem. Suppose that a uniform square sheet of metal has two opposite sides maintained at the freezing point of water ($0^{\circ}\text{C}.$) while the other two sides are maintained at the boiling point of water ($100^{\circ}\text{C}.$). Suppose further that no heat is conducted away except along the edges, and that the *additivity* of states of temperature is assumed. It is asked, on the basis of these very scanty assumptions: what are the permanent temperatures set up along the two diagonals?



The Principle of Sufficient Reason alone suffices to show that along these two lines the temperature must be everywhere exactly $50^{\circ}\text{C}.$ In fact, rotate the square by half a revolution about the diagonal AC (see the above figure), so that pairs of adjacent sides are interchanged while the

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individual points of the diagonal AC remain fixed. By the additivity assumed, the resultant combined temperature will be 100° along all of the sides and double the temperature along the fixed diagonal. But if the temperature is permanently 100°C. along the boundary, it will be 100°C. everywhere inside, so that the temperature all along the diagonal AC *must* have been exactly 50°C. ; and the same must be true along the other diagonal BD .

Evidently this is a remarkable conclusion, made on the basis of limited assumptions without any calculation! This is possible because of the peculiar symmetry of the problem which permits the use of the underlying group of rotations of the square in order to arrive at a precise quantitative result.

These illustrations are sufficient to show the exceedingly great suggestiveness of the Principle of Sufficient Reason for physics. It has been used, more or less tacitly, in almost all classical physics of the Newtonian type.

However, in the subsequent electromagnetic era inaugurated by Faraday and Maxwell, it seemed until the beginning of the present century as though the Principle of Sufficient Reason had no important part to play. In fact, the ordinary group of motions in no way suggested the peculiar laws interrelating the electric and magnetic lines of force visualized by Faraday. Here one was apparently confronted by mysterious entities of non-mechanical nature which were entirely foreign to ordinary physical intuition.

It is therefore a striking fact that in 1908 the mathematician Minkowski found a more natural way of looking at the phenomena of electromagnetism. He discovered that by introducing as basic a certain "Lorentz" group instead of the group of motions, the observed laws lost their apparent artificiality and took on a character of inevitability from a

higher, group-theoretic point of view. Minkowski foresaw clearly the importance of this forward step, made shortly before his death.

The Lorentz group is applicable to a (3+1)-dimensional space-time, and is made up of operations which mix up the three space coördinates and the single time coördinate of two systems in relative motion. At low velocities compared to that of light, the quantitative difference between the group of Euclidean motions¹¹ and the Lorentz group is insignificant.

The point of view of Minkowski is consonant with that of the special theory of relativity of Einstein (1905) and has been universally adopted in dealing with electromagnetism. It permits us to write in a quite legitimate way the mystical equation

$$186,300 \text{ miles} = \sqrt{-1} \text{ seconds.}$$

It also enables us to answer in similar fashion the question: Why is space three-dimensional, and why is time one-dimensional? The answer is that a theory of electromagnetic type can be proved to be possible only in the case of a three-dimensional space and a one-dimensional time!

Here it is interesting to recall that Leibniz, as a systematic philosopher, considered that the fact that space was three-dimensional followed as a geometric necessity from the usual axioms of geometry, for he says: "The triple number of dimensions is determined, not by the reason of the best [i.e., not by a special application of the Principle of Sufficient Reason], but by a geometrical necessity: it is because geometers have been able to show that there are only three mutually perpendicular straight lines which can intersect in the same point." Here of course Leibniz was entirely

¹¹Together with a change in origin of time.

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wrong, since n -dimensional Euclidean geometry is just as possible as three-dimensional. However, if he were living today, he would undoubtedly “explain” the triple dimensionality of space on the electromagnetic basis outlined above.

A somewhat analogous situation arises for number systems of so-called non-commutative type, different from ordinary numbers and yet permitting of the operation of division: here the only type is essentially that of the four-dimensional quaternions discovered by the Irish mathematician William Rowan Hamilton.

It was Minkowski's paper of 1908 which stimulated Einstein to formulate his general gravitational theory of 1915. In this theory, too, the Principle of Sufficient Reason played a notable part, for it was the complete directional ambiguity of space about the central Sun which alone enabled Einstein to draw his conclusions. He did not explicitly appeal to the Principle, but it was used just the same. As a matter of fact, the gravitational theory of relativity has so far only been successfully applied in this very special, highly symmetrical case of a large spherical mass.

In passing it may be observed that Einstein's general theory of relativity in its first form can itself effectively be criticized on the basis of the same Principle of Sufficient Reason! In fact, if there were only a single central body, this body might be rotating *according to that theory*. But with respect to what could it rotate?

It is remarkable that Leibniz himself should have been led to take a specifically relativistic point of view towards space and time. No doubt this suited him because it stood in opposition to the absolutistic point of view of his great rival Newton. But Leibniz was led to adopt this position primarily through the Principle of Sufficient Reason, as the

following quotation shows: "I hold space, and also time, to be something purely relative. Space is an *order of coexistences* as time is an *order of successions*. Space denotes in terms of possibility an order of things which, in so far as they exist together, exist at the same time, whatever be their several ways of existing." Again he says, "There are many ways of refuting the imagination of those who take space to be a substance, or at least something absolute. I say that, if space were an absolute Being, it would be impossible to give a sufficient reason for anything that might happen, yet this principle is with us an axiom." He then goes on to give a proof of his assertion, based on the indistinguishability of different points of ideal space and of different instants of ideal time, which makes it inconceivable that Deity selects any one point or instant in preference to others. Here evidently it is the ambiguity of space corresponding to the group of motions, and the ambiguity of time corresponding to the group of time measurements, that brings in the Principle.

Thus once more a great philosopher has revealed a prophetic insight; and it may be added that the French philosopher Bergson was led on purely philosophical grounds to formulate the (relativistic) idea of "local time" even before Lorentz and Einstein were led to do so on the basis of the celebrated Michelson-Morley experiment and the equations of Maxwell.

In the latest quantum-mechanical phase of physics, the attempt to maintain a conceptual grasp of physical law has been at least temporarily abandoned in favor of a purely formal attack by means of mathematical guesswork. This process, it must be admitted, has so far been strikingly successful. Nevertheless I believe the day not to be far distant when less artificial methods must again be employed.

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If before Newton's time the laws of Kepler for the motion of two bodies had been used as a basis for similar shrewd conjectures concerning the three-body problem presented by the Sun, Earth, and Moon, the proper formulas would probably have been obtained; for there were always the known observational facts to guide one *ad hoc* when in difficulty. It is such an *ad hoc* "explanation" that quantum-mechanics seems to me to provide for the treatment of spectroscopic and allied phenomena. Just as Newton's law of gravitation, of conceptual type and largely based on the Principle of Sufficient Reason, would have displaced such an artificial treatment of the three-body problem, so we may expect an alternative conceptual approach to quantum-mechanical laws to be found and adopted.

In the biological and psychological domains there is as yet little or no occasion to use the Principle of Sufficient Reason, except in the extremely vague way which we proceed to indicate.

Physicists have always treated matter as mere dead stuff, although, from the philosophic point of view, matter must be thought of as the potential abode of life. This tacit hypothesis of the physicists is probably a useful one in our present stage of physical knowledge. But an unfortunate effect of it has been to make matter seem alien to life, although recently there are signs that the attitude of biologists is changing in this respect.

However, Leibniz avoided this error of thought, for his theory of monads led him, at least in the early stages of his philosophical development, to locate souls in points, death itself being merely the contraction of a monad dominating an "entelechy" to a single point. The Leibnizian employment of the Aristotelian concept of entelechy in order to explain the activity of living organisms may be regarded as an

important biological advance. Such an entelechy was based on the concept of a dominant monad in a monadic system. Leibniz envisaged the existence of continuous series of such monadic beings. These may be illustrated not only by cells and other organisms, but by insect communities, and by other forms as yet unknown. Leibniz said in this connection "I definitely avow that there are in the world animals as much larger than ourselves, as we are larger than microscopic animalcules." One is reminded in this connection of the incident related by his commentator Hantschius, that Leibniz once remarked to him over a cup of coffee, "There may be in this cup a monad which will one day be a rational soul!"

Of course the existence of monads was "deduced" from his fundamental Principles of Contradiction and Sufficient Reason, but it is difficult to see the cogency of Leibniz' vitalistic arguments.

As far as this ever-recurring conflict between mechanistic and vitalistic points of view among biologists is concerned, it seems to me that it may be at once decided in favor of the vitalists by a reasonable philosophic use of the Principle of Sufficient Reason: *for, no sufficient reason for the observed behavior of living organisms is to be found in purely mechanical systems.* For example, from a purely mathematical point of view a system of Newtonian type would necessarily exhibit a type of eternal Nietzschean recurrence, foreign to living things!

In the fivefold hierarchy of the levels of knowledge—mathematical, physical, biological, psychological, social—the social level is of course the most complex and difficult of all. Nevertheless it is already clear that the Principle of Sufficient Reason is likely to play a definite rôle in certain technical problems.

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My first illustration is taken from the question of the proper apportionment among the several states of the assigned total of Representatives in Congress (referred to in the preceding lecture). Huntington has emphasized the fact that the prevailing Willcox system of apportionment is based on the individual share, n/N (N = population, n = number of Representatives), whereas there is an ignored but symmetric system based on the idea of the Congressional district N/n . Huntington's own system is of an intermediate type, favoring neither the Congressional district nor individual share. It seems obvious then that, from the point of view of our Principle as it would have been interpreted by Leibniz, the Huntington system is to be logically preferred to either of the other two systems.

My second illustration will be taken from the field of the law, and I am much indebted to my colleague Roscoe Pound for supplying me with it. The case presented is of the following general character. Two small groups of individuals, A and B , request that electric light facilities be extended from the town to their respective nearby communities, the cost of this extension to be divided between them. How should the cost be allocated? As I understand it, the accepted method would be as follows: All of the interests involved and reasonable ways of distributing the cost would be considered, and then a compromise would be struck between these. For example, the basis might be the number of individuals served, or the amount of electricity to be used, or the costs of installation separately, etc. All these, in so far as really different and equally valid, would receive the same consideration in the final decision. It is assumed of course that the case has not been already disposed of by earlier precedents. Evidently this is the proper method of solution on the basis of our Principle.

These two examples suggest that, in the systematic treatment of ethical questions, the Principle of Sufficient Reason may sometimes afford the best way out of an otherwise insoluble problem.

In conclusion I would like to mention one major philosophic result to which the Principle of Sufficient Reason seems to point. It is closely allied with Leibniz' own conclusion that this is the best of all possible worlds, but in reality quite distinct from it. Leibniz elaborates this conclusion in his statement: "It follows from the supreme perfection of God that in producing the universe he has chosen the best possible plan, the greatest variety combined with the most perfect order; ground, place, times arranged as well as possible; the maximum effect received by the simplest means; as much power, knowledge, well-being and goodness in creatures as the universe would admit. . . . Otherwise, it would be impossible to find a reason why things are thus rather than otherwise." In Leibniz's feeling that the root-notions will be found to be few in number, he comes still closer to the idea which I have in mind: says Leibniz, "I believe there are only a few primitive decrees which regulate the consequences of things." Such principles as that of "least action" in dynamics, discovered independently by Leibniz and Maupertuis, seemed to Leibniz to be corroborative of this mystical truth.

Another aspect of the same idea was tacitly employed by Newton when he selected the second power of the distance in formulating his gravitational law, rather than some very nearby power. It may be remarked that such quasi-aesthetic feelings constantly guide the efforts of any successful system-builder in science or philosophy.

Einstein, too, has expressed a variation of the same theme in terms which are perhaps intended to be theological only

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in an allegorical sense. He declares: "*Raffiniert ist der Herr Gott, boshaft ist Er nicht*"—God is subtle, He is not malicious.

Very recently an analogous thesis has been advanced by Sir Arthur Eddington, seeming at first sight to deny the necessity of experimentation. Says Eddington: "Unless the structure of the nucleus has a surprise in store for us, the conclusion seems plain—there is nothing in the whole system of laws of physics that cannot be deduced unambiguously from epistemological considerations. An intelligence, unacquainted with our universe but acquainted with the system of thought by which the human mind interprets to itself the content of its sensory experience, should be able to attain all the knowledge of physics that we have attained by experiment. . . . For example, he would infer the existence and properties of radium, but not the dimensions of the earth."

In making this statement, Eddington apparently overlooks the fact that today we are surrounded by instruments of precision; in other words all of us live in a veritable scientific laboratory, and thus acquire easily an understanding of certain facts of nature which the keenest observer could not have guessed at in ancient times. Eddington regards theories thus arrived at as essentially subjective, but I do not see how this alters the basic significance of his mystical conclusion.

All of these somewhat different points of view indicate a prevailing faith that the Order of Nature is supernal, combining logical simplicity and inevitability in a most remarkable way. Ordinary geometry affords such an account of space; Newtonian dynamics, of the behavior of matter in motion; electromagnetism, of the facts of light and electricity; etc. This belief is grounded, on the one hand, in scientific experience and, on the other, in the successful use

of the Principle of Sufficient Reason in more or less metaphysical form.

Up to the present point my aim has been to consider a variety of applications of the Principle of Sufficient Reason, without attempting any precise formulation of the Principle itself. With these applications in mind I will venture to formulate the Principle and a related Heuristic Conjecture in quasi-mathematical form as follows:

PRINCIPLE OF SUFFICIENT REASON. If there appears in any theory T a set of ambiguously determined (i.e. symmetrically entering) variables, then these variables can themselves be determined only to the extent allowed by the corresponding group G . Consequently any problem concerning these variables which has a uniquely determined solution, must itself be formulated so as to be unchanged by the operations of the group G (i.e. must involve the variables symmetrically).

HEURISTIC CONJECTURE. The final form of any scientific theory T is: (1) based on a few simple postulates; and (2) contains an extensive ambiguity, associated symmetry, and underlying group G , in such wise that, if the language and laws of the theory of groups be taken for granted, the whole theory T appears as nearly self-evident in virtue of the above Principle.

The Principle of Sufficient Reason and the Heuristic Conjecture, as just formulated, have the advantage of not involving excessively subjective ideas, while at the same time retaining the essential kernel of the matter.

In my opinion it is essentially this principle and this conjecture which are destined always to operate as the basic criteria for the scientist in extending our knowledge and understanding of the world.

It is also my belief that, in so far as there is anything definite in the realm of Metaphysics, it will consist in further applications of the same general type. This general conclu-

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sion may be given the following suggestive symbolic form:

Metaphysics \longleftrightarrow Principle of Sufficient Reason \longleftrightarrow Theory of Ambiguity and Groups.

While the skillful metaphysical use of the Principle must always be regarded as of dubious logical status, nevertheless I believe it will remain the most important weapon of the philosopher.

NOTE ON THE LAW OF THE PARALLELOGRAM OF FORCES

In order to show how this law may be regarded as a kind of consequence of the Principle of Sufficient Reason, we begin by laying down the following three postulates:

(I) Collinear vector forces at a point O have a resultant force in the same line whose magnitude is the (algebraic) sum of the magnitudes of the two constituent forces.

(II) Any two forces have a unique resultant. The combination of forces into resultants is associative and commutative:

$$(F+G)+H=F+(G+H);$$

$$F+G=G+F.$$

The resultant of forces varies continuously with the constituent forces.

(III) *Principle of Sufficient Reason.* The resultant of two forces is independent of the choice of axes of reference and of the unit of force.

It is easy to show on the basis of these postulates that the law of composition of forces is simply the law of vector addition, embodied in the so-called Parallelogram Law.

To do so we remark first that the resultant H of two non-collinear forces F and G cannot lie in the line of either of these forces. For from

$$H=F+G$$

we obtain by (II) and (I)

$$(-F) + H = ((-F) + F) + G = G.$$

Hence if H and $(-F)$ are collinear, G will be collinear with them by I , which is impossible. Thus H and F (and similarly H and G) cannot be collinear.

We also observe that the unique resultant H of two forces F and G (see II) must lie in their plane by the Principle of Sufficient Reason (III); for why should this resultant lie on one side of this plane rather than the other?

Now let us select two orthogonal axes in any plane through O and consider forces X and Y along these axes in the chosen positive sense, with magnitudes $\cos \alpha$ and $\sin \alpha$ respectively where α ranges from 0 to $\pi/2$. For $\alpha=0$ the resultant is a unit vector along the x -axis; for $\alpha=1$ the resultant is a unit vector along the y axis. As α varies from 0 to $\pi/2$ the resultant vector OP varies continuously by (I). Thus P describes a continuous curve (see figure 1).

It is clear that this curve can not intersect the x or y axis, for $0 < \alpha < \pi/2$, by our first remark, and so must lie wholly within the first quadrant as indicated in the figure. Consequently OP must take any desired direction in the quadrant at least once, and, since the resultant by the same Principle is independent of the unit selected we may decompose any force H whose direction falls in this quadrant into two such forces X and Y ,

$$H = X + Y.$$

But there cannot be more than one such decomposition, for if

$$X + Y = X' + Y',$$

then

$$(X - X') + (Y - Y') = 0;$$

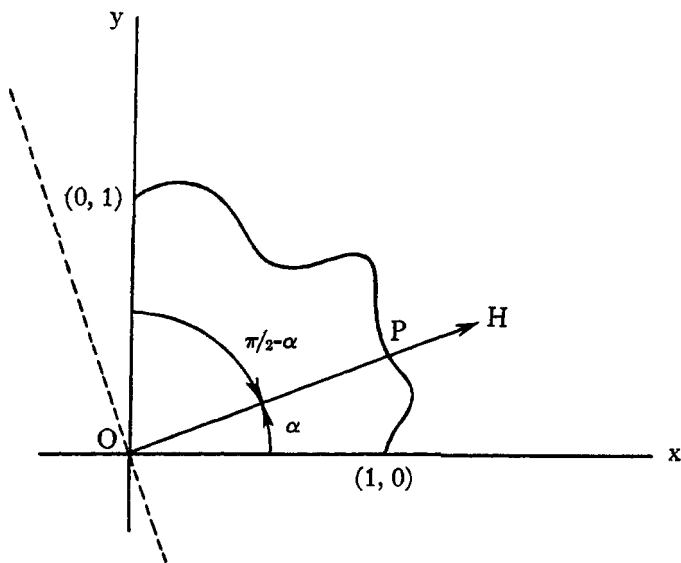


FIG. 1.

whence by our first remark

$$X = X', \quad Y = Y'.$$

Furthermore we see by the intrinsic geometry of the same figure, based upon our Principle, that

$$(1) \quad X = h\varphi(\alpha) \quad Y = h\varphi(\pi/2 - \alpha)$$

where h and α designate the magnitude and angle of H , and φ is a continuous function of α such that

$$\varphi(0) = 1, \quad \varphi(1) = 0, \quad 0 < \varphi(\alpha) \text{ for } 0 < \alpha < \pi/2.$$

Here we have let X and Y stand (ambiguously) for the magnitudes of the vectors X and Y of specified directions.

Next let us resolve X and Y along the direction of H and the perpendicular direction (see the same figure). Clearly we obtain

$$\begin{cases} h = X\varphi(\alpha) + Y\varphi(\pi/2 - \alpha) \\ 0 = -X\varphi(\pi/2 - \alpha) + Y\varphi(\alpha). \end{cases}$$

The second of these relations reduces to an identity in virtue of (1) but the first yields by combination with (1)

$$(2) \quad \varphi^2(\alpha) + \varphi^2(\pi/2 - \alpha) = 1,$$

and so in particular $\varphi(\pi/4) = 1/\sqrt{2}$.

This shows that we have always

$$(3) \quad h^2 = X^2 + Y^2$$

in accordance with the Parallelogram Law.

To complete our deduction of that Law we now consider two forces of equal magnitude h making an angle α ($0 < \alpha < \pi/2$) with one another as in figure 2.

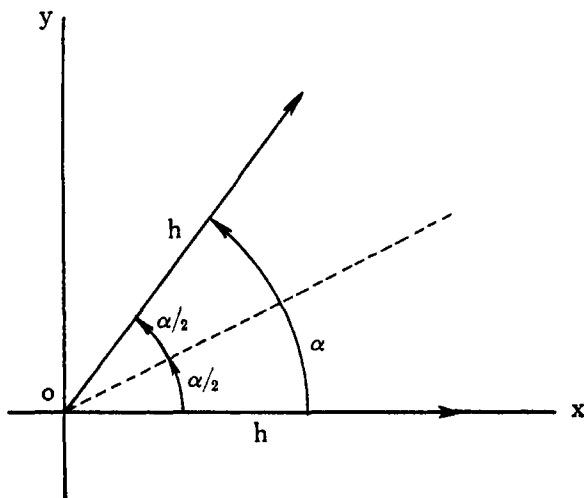


FIG. 2.

We first resolve both of these forces along the line of one of them and in a perpendicular direction obtaining components.

$$h + h\varphi(\alpha) \quad \text{and} \quad h\varphi(\pi/2 - \alpha)$$

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respectively. By (3) and (2) the resultant is therefore of magnitude

$$\{[h + h\varphi(\alpha)]^2 + h^2\varphi^2(\pi/2 - \alpha)\}^{\frac{1}{2}} = h[2(1 + \varphi(\alpha))]^{\frac{1}{2}}.$$

Secondly we resolve along the bisector of the lines of the forces and perpendicularly to this bisector, obtaining components

$$2h\varphi(\alpha/2) \quad \text{and} \quad 0$$

respectively. Hence we conclude that

$$(4) \quad \varphi^2(\alpha/2) = \frac{1 + \varphi(\alpha)}{2}, \quad 0 < \alpha < \pi/2.$$

From the companion formulas (2) and (4) we conclude that for all α of the form $m\pi/2^n$ we have $\varphi(\alpha) = \cos \alpha$. In fact we already have found that

$$\varphi(0) = 1, \quad \varphi(\pi/2) = 0, \quad \varphi\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}};$$

and by bisection we find next

$$\varphi\left(\frac{\pi}{8}\right) = \cos \pi/8, \quad \varphi\left(\frac{3\pi}{8}\right) = \cos \frac{3\pi}{8};$$

and so forth. But $\varphi(\alpha)$ is continuous and so we see that for all α we have $\varphi(\alpha) = \cos \alpha$. Thus (1) takes the form

$$(5) \quad X = h \cos \alpha, \quad Y = h \sin \alpha$$

in accordance with the Parallelogram Law of resolution along these axes. Furthermore it is clear that the formula (5) holds for any α whatsoever. Hence it follows that any two forces H_1 and H_2 combine in the desired manner since we have

$$X_1 = h_1 \cos \alpha_1, \quad Y_1 = h_1 \sin \alpha_1; \quad X_2 = h_2 \cos \alpha_2, \quad Y_2 = h_2 \sin \alpha_2,$$

with resultant components,

$$h_1 \cos \alpha_1 + h_2 \cos \alpha_2, \quad h_1 \sin \alpha_1 + h_2 \sin \alpha_2,$$

in the selected directions.